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ABSTRACT - In order to compute economic equilibria in an exchange economy by means of a simplicial algorithm, we propose a new simplicial subdivision of the underlying price space, being the unit simplex. We call this subdivision the T_1'-triangulation. We show that the T_1'-triangulation is superior to the well known K_1'- and J_1'-triangulations according to measures of efficiency of triangulations.

1. INTRODUCTION

Since van der Laan and Talman [8] proposed the first simplicial variable dimension algorithm without an extra dimension, a lot of contributions to this field have appeared, e.g., see [3], [4], [6], [8], and [11]. These algorithms can be used to find an economic equilibrium in a pure exchange economy. They subdivide the price space, being the n-dimensional unit simplex S^n, into simplices and search for a simplex that yields an approximate equilibrium. So far, we might say that the triangulations which underly all these algorithms are adaptions or generalizations to S^n of the well known K_1- and J_1-triangulations of $\mathbb{R}^n$. In [2], we proposed a new simplicial subdivision of $\mathbf{R}^n$, called the D_1-triangulation, which is superior to the K_1- and J_1-triangulations. In order to improve the algorithms on S^n, we want to construct a simplicial subdivision of S^n by using the D_1-triangulation, which is superior to the other triangulations of S^n.

Section 2 introduces the new triangulation of S^n, called the T_1'-triangulation. We give the pivot rules of the T_1'-triangulation in section 3. It is compared with the K_1'- and J_1'-triangulations in section 4.

2. A NEW TRIANGULATION OF THE UNIT SIMPLEX

Let m be a positive integer. Let $C^n(m) = \{x \in \mathbf{R}^n \mid m \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}$. Clearly, $C^n(m)$ is an n-dimensional simplex in $\mathbf{R}^n$. We first triangulate $C^n(m)$ into n-dimensional simplices and then transform $C^n(m)$ to the n-dimensional unit simplex $S^n = \{x \in \mathbf{R}^{n+1} \mid \sum_{j=1}^{n+1} x_j = 1$, and $x_j \geq 0$ for $j = 1, \ldots, n+1\}$. For $h = 1, \ldots, n$, let u^h denote the h-th unit vector in $\mathbf{R}^n$. Let $T_1^{0c} = \{y \in C^n(m) \mid$ all components of y are odd$\}$. Let $y_0 = m$ and $y_{n+1} = 0$. Let $y \in T_1^{0c}$. Let

$$I = I(y) = \{i \mid y_{i+1} < y_i < y_{i-1}\},$$

$$I^+ = I^+(y) = \{i \mid y_{i+1} = y_i < y_{i-1}\},$$

$$I^- = I^-(y) = \{i \mid y_{i+1} < y_i = y_{i-1}\},$$

and $J = J(y) = \{i \mid y_{i+1} = y_i = y_{i-1}\}$.

Let $N = \{1, 2, \ldots, n\}$. It is obvious that for each $y \in T_1^{0c}$,

$$N = I(y) \cup J(y) \cup I^+(y) \cup I^-(y).$$

Let $s = (s_1, s_2, \ldots, s_n)^T$ be a sign vector such that $s_i \in \{-1, +1\}$ for $i = 1, 2, \ldots, n$. Let $T = T(y, s) = \{i \mid i \in I$, or $i \in I^+$ and $s_i = 1$, or $i \in I^-$ and $s_i = -1\}$. Let π be a permutation of the elements of N. Let $q = q(\pi)$ be a nonnegative integer such that $\pi(i) \in T$ for $i = n-q+1, \ldots, n$ and $\pi(n-q) \notin T$. Let $0 \le p \le q-1$ be an integer. Assume $n \ge 2$.

If $q = n$ and $p = 0$, let $y^0 = y$,

$$y^k = y + s_{\pi(k)}u^{\pi(k)},\ k = 1, 2, \ldots, n.$$

If $q = n$ and $p \ge 1$, let $y^0 = y + s$,

$$y^k = y^{k-1} - s_{\pi(k)}u^{\pi(k)},\ k = 1, 2, \ldots, p-1,$$

$$y^k = y + s_{\pi(k)}u^{\pi(k)},\ k = p, \ldots, n.$$

When $2 \le q < n$, let $y^0 = y + s$,

$$y^k = y^{k-1} - s_{\pi(k)}u^{\pi(k)},\ k = 1, 2, \ldots, n-q-1,$$

and if $p = 0$, let $y^{n-q} = y$,

$$y^k = y + s_{\pi(k)}u^{\pi(k)},\ k = n-q+1, \ldots, n,$$

and if $p \ge 1$, let

$$y^k = y^{k-1} - s_{\pi(k)}u^{\pi(k)},\ k = n-q, \ldots, n-q+p-1,$$

$$y^k = y + s_{\pi(k)}u^{\pi(k)},\ k = n-q+p, \ldots, n.$$

When $q < 2$, let $y^0 = y + s$,

$$y^k = y^{k-1} - s_{\pi(k)}u^{\pi(k)},\ k = 1, 2, \ldots, n.$$

Let $y^0, y^1, \ldots, y^n$ be produced in the above manner, then it is obvious that they are affinely independent. Let $\sigma = [y^0, y^1, \ldots, y^n]$ be the convex hull of $y^0, y^1, \ldots, y^n$. Thus σ is an n-dimensional simplex, which will be denoted by $T_1(y, \pi, s, p, q)$. Let $T_1(m)$ be the set of all such simplices $T_1(y, \pi, s, p, q)$ in $C^n(m)$.

Lemma 2.1. $\bigcup_{\sigma\in T_1(m)} \sigma = C^n(m)$.

Proof. Let $x \in C^n(m)$. Let

$$y_i = \begin{cases} \lfloor x_i \rfloor + 1, & \text{if } \lfloor x_i \rfloor \text{ is even and } \lfloor x_i \rfloor < m, \\ \lfloor x_i \rfloor - 1, & \text{if } \lfloor x_i \rfloor \text{ is even and } \lfloor x_i \rfloor = m, \\ \lfloor x_i \rfloor, & \text{otherwise,} \end{cases}$$

and let

$$s_i = \begin{cases} -1, & \text{if } \lfloor x_i \rfloor \text{ is even and } \lfloor x_i \rfloor < m \text{ or } \lfloor x_i \rfloor \text{ is odd and } \lfloor x_i \rfloor = m, \\ 1, & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$. Obviously, $y = (y_1, y_2, \ldots, y_n)^\top \in T_1^{0c}$. Let π be a permutation of N such that $0 \le s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}) \le s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) \le \ldots \le s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)}) \le 1$, where in case $s_{\pi(i)}(x_{\pi(i)} - y_{\pi(i)}) = s_{\pi(j)}(x_{\pi(j)} - y_{\pi(j)})$ and $i < j$, then $\pi(i) < \pi(j)$ if $s_{\pi(i)} = -1$, and $\pi(i) > \pi(j)$ if $s_{\pi(i)} = 1$. Let $q = q(\pi)$.

When $q < 2$, let $y^0 = y + s$, $y^k = y^{k-1} - s_{\pi(k)}u^{\pi(k)}$, $k = 1, 2, \ldots, n$, and let $q_0 = s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)})$, $q_1 = s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) - s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)})$, $\ldots$, $q_{n-1} = s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)}) - s_{\pi(n-1)}(x_{\pi(n-1)} - y_{\pi(n-1)})$, $q_n = 1 - \sum_{k=0}^{n-1} q_k = 1 - s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)})$. Obviously, $\sum_{k=0}^{n} q_k = 1$, $q_k \ge 0$ for all k, and $x = \sum_{k=0}^{n} q_k y^k$. So $x \in \sigma = [y^0, y^1, \ldots, y^n] \subseteq C^n(m)$.

When $q = n$, the proof is similar to that of Lemma 2.2 in [2].

Suppose $2 \le q < n$. If

$$\sum_{k=n-q}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - q s_{\pi(n-q)}(x_{\pi(n-q)} - y_{\pi(n-q)}) \le 1,$$

let $q_0 = s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)})$, $q_1 = s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) - s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)})$, ..., $q_{n-q-1} = s_{\pi(n-q)}(x_{\pi(n-q)} - y_{\pi(n-q)}) - s_{\pi(n-q-1)}(x_{\pi(n-q-1)} - y_{\pi(n-q-1)})$, $q_{n-q} = 1 - (\sum_{k=n-q}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - qs_{\pi(n-q)}(x_{\pi(n-q)} - y_{\pi(n-q)}))$, $q_{n-q+1} = s_{\pi(n-q+1)}(x_{\pi(n-q+1)} - y_{\pi(n-q+1)}) - s_{\pi(n-q)}(x_{\pi(n-q)} - y_{\pi(n-q)})$, ..., $q_n = s_{\pi(n)}(x_{\pi(n)} - y_{\pi(n)}) - s_{\pi(n-q)}(x_{\pi(n-q)} - y_{\pi(n-q)})$, $p = 0$, $y^0 = y + s$, $y^k = y^{k-1} - s_{\pi(k)}u^{\pi(k)}$, $k = 1, 2, \ldots, n-q-1$, $y^{n-q} = y$, $y^k = y + s_{\pi(k)}u^{\pi(k)}$, $k = n-q+1, \ldots, n$, then $q_k \geq 0$ for all k, $\sum_{k=0}^{n} q_k = 1$, and $x = \sum_{k=0}^{n} q_k y^k$. Thus $x \in \sigma = [y^0, y^1, \ldots, y^k] \subseteq C^n(m)$. If

$\sum_{k=n-q}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - qs_{\pi(n-q)}(x_{\pi(n-q)} - y_{\pi(n-q)}) > 1$,

we show that there exists an integer $1 \leq p \leq q-1$ such that the following system has a nonnegative solution,

$$q_0 = s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}),$$

$$q_1 = s_{\pi(2)}(x_{\pi(2)} - y_{\pi(2)}) - s_{\pi(1)}(x_{\pi(1)} - y_{\pi(1)}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$q_{n-q+p-2} = s_{\pi(n-q+p-1)}(x_{\pi(n-q+p-1)} - y_{\pi(n-q+p-1)}) - s_{\pi(n-q+p-2)}(x_{\pi(n-q+p-2)} - y_{\pi(n-q+p-2)}),$$

$$q_{n-q+p-1} = -s_{\pi(n-q+p-1)}(x_{\pi(n-q+p-1)} - y_{\pi(n-q+p-1)}) + (\textstyle\sum_{k=n-q+p}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - 1)/(q-p),$$

$$q_k = s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) + (1 - \textstyle\sum_{k=n-q+p}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}))/(q-p), \quad k = n-q+p, \ldots, n.$$

If $q_{n-q+p-1} \geq 0$ for $p = q-1$, it is clear that $q_k \geq 0$ for all k. If not, then since $\sum_{k=n-q}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - qs_{\pi(n-q)}(x_{\pi(n-q)} - y_{\pi(n-q)}) > 1$, there exists an integer $1 \leq p_0 \leq q-2$ such that

$-s_{\pi(n-q+p_0-1)}(x_{\pi(n-q+p_0-1)} - y_{\pi(n-q+p_0-1)}) + (\sum_{k=n-q+p_0}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - 1)/(q-p_0) \geq 0,$

$-s_{\pi(n-q+p_0)}(x_{\pi(n-q+p_0)} - y_{\pi(n-q+p_0)}) + (\sum_{k=n-q+p_0+1}^{n} s_{\pi(k)}(x_{\pi(k)} - y_{\pi(k)}) - 1)/(q-p_0-1) < 0.$

Hence, $s_{\pi(n-q+p_0)}(x_{\pi(n-q+p_0)} - y_{\pi(n-q+p_0)}) + (1 - \sum_{k=n-q+p_0}^{n} s_{\pi(k)} (x_{\pi(k)} - y_{\pi(k)}))/(q-p_0) \geq s_{\pi(n-q+p_0)}(x_{\pi(n-q+p_0)} - y_{\pi(n-q+p_0)}) + (- (q-p_0-1)\, s_{\pi(n-q+p_0)}(x_{\pi(n-q+p_0)} - y_{\pi(n-q+p_0)}) - s_{\pi(n-q+p_0)}(x_{\pi(n-q+p_0)} - y_{\pi(n-q+p_0)}))/(q-p_0) = 0$. Thus when $p = p_0$, $q_k \geq 0$ for all k. Obviously, $\sum_{k=0}^{n} q_k = 1$. Let $y^0 = y + s$, $y^k = y^{k-1} - s_{\pi(k)}u^{\pi(k)}$, $k = 1, 2, \ldots, n-q+p-1$, and $y^k = y + s_{\pi(k)}u^{\pi(k)}$, $k = n-q+p, \ldots, n$. We easily obtain that $x = \sum_{k=0}^{n} q_k y^k$. Therefore, $x \in \sigma = [y^0, y^1, \ldots, y^n] \subseteq C^n(m)$.

From the above conclusions, the lemma follows immediately.

<u>Theorem 2.2</u>. $T_1(m)$ is a triangulation of $C^n(m)$.

<u>Proof</u>. From Lemma 2.1 and the definition, the theorem follows immediately.

We call the triangulation $T_1(m)$ the T_1-triangulation of $C^n(m)$. Let P be the (n+1)xn matrix defined by

$$P = \begin{bmatrix} -1 & & & & \\ 1 & -1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & -1 \\ & & & \cdot & 1 \end{bmatrix}$$

and let $\bar{u}^1 = (1, 0, \ldots, 0)^T \in R^{n+1}$. Clearly, $S^n = m^{-1}PC^n(m) + \{\bar{u}^1\}$. Let $T_1' := m^{-1}PT_1 + \{\bar{u}^1\} = \{m^{-1}P\sigma + \{\bar{u}^1\} \mid \sigma \in T_1\}$.
Then T_1' is a triangulation of S^n. It is called the T_1'-triangulation with grid size m^{-1}. For $n = 2$ and $m = 4$ and for $n = 3$ and $m = 2$ the T_1'-triangulation of S^n with grid size m^{-1} is illustrated in the figures 1 and 2, respectively.

3. THE PIVOT RULES OF THE T_1'-TRIANGULATION

For convenience, we consider the pivot rules of the T_1-triangulation of $C^n(m)$.

Let $\sigma = [y^0, y^1, \ldots, y^n] = T_1(y, \pi, s, p, q)$ be given. We want to obtain the parameters of $\bar{\sigma} = [\bar{y}^0, \bar{y}^1, \ldots, \bar{y}^n] = T_1(\bar{y}, \bar{\pi}, \bar{s}, \bar{p}, \bar{q})$ such that all vertices of σ are its vertices except y^i, in case the facet of σ opposite to vertex y^i does not lie in the boundary of $C^n(m)$. In Table (3-1), we show how $\bar{y}$, $\bar{\pi}$, $\bar{s}$, $\bar{p}$, and $\bar{q}$ are determined from y, π, s, p, q, and i. From this table, it is easy to obtain each vertex $\bar{y}^k$ of $\bar{\sigma}$, $k = 0, 1, \ldots, n$, and in particular its new vertex.

4. THE COMPARISON OF THE K_1'-, J_1'-, AND T_1'-TRIANGULATIONS OF S^n

In [2], we have introduced the D_1-triangulation of $\mathbb{R}^n$ and demonstrated that it is superior to the well known K_1- and J_1-triangulations (see [1], [5], [10]) according to the number of simplices in the unit cube, the diameter, the average direction density, and the surface density. From the definition of the T_1-triangulation of $C^n(m)$, it can be seen that we use the D_1-triangulation to triangulate the interior of $C^n(m)$ and the J_1-triangulation to triangulate the boundary of $C^n(m)$, i.e., we obtain the T_1-triangulation of $C^n(m)$ by combining the D_1-triangulation and the J_1-triangulation. Hence, the T_1-triangulation is superior to the triangulations of $C^n(m)$, which are obtained from the K_1- and J_1-triangulations. Therefore, the T_1'-triangulation is superior to the well known K_1'- and J_1'-triangulations (see [9] and [10]) of S^n according to the same measures, where the K_1'- and J_1'-triangulations have been obtained from the K_1- and J_1-triangulations in the same way as the T_1'- from the T_1-triangulation.

Figure 1. T_1' - triangulation of S^2 with $m^{-1} = 1/4$.

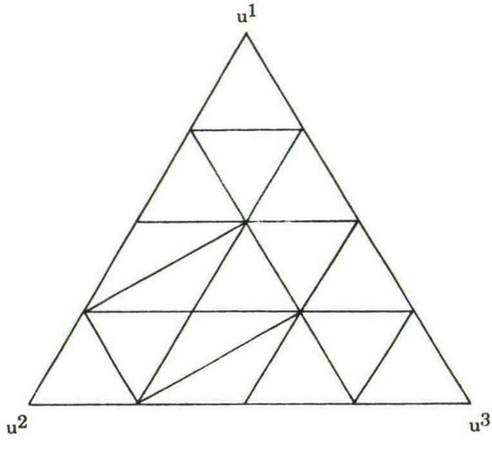

Figure 2. T_1' - triangulation of S^3 with $m^{-1} = 1/2$

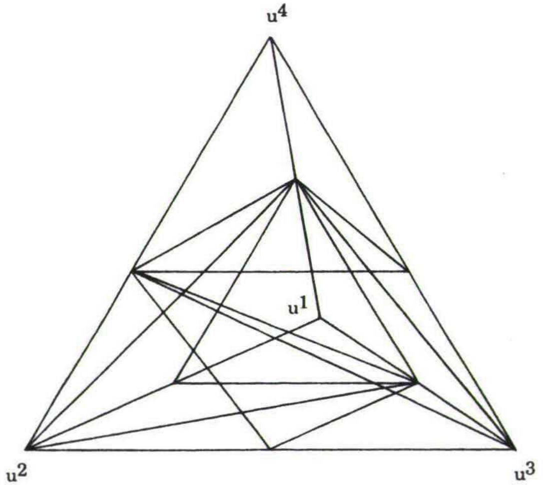

Table (3-1) The Pivot Rules of the T_1-Triangulation

i	p	q	π	$\bar{y}$	$\bar{s}$	$\bar{\pi}$	$\bar{p}$	$\bar{q}$
0	0	n		y	s	π	$p+1$	q
0	1	n		y	s	π	$p-1$	q
0	$2\le p\le q-1$	n	$\pi(1)\in I^{+}(y)\cup I^{-}(y)$	y	$s-2s_{\pi(1)}u^{\pi(1)}$	π	$p-1$	$q-1$
0	$2\le p\le q-1$	n	$\pi(1)\notin I^{+}(y)\cup I^{-}(y)$	y	$s-2s_{\pi(1)}u^{\pi(1)}$	π	p	q
0	0	$n-1$	$\pi(1)\in J(y)$	y	$s-2s_{\pi(1)}u^{\pi(1)}$	π	p	q
0	0	$n-1$	$\pi(1)\in I^{+}(y)\cup I^{-}(y)$	y	$s-2s_{\pi(1)}u^{\pi(1)}$	π	p	$q+1$
0	$p\ge1$	$n-1$	$\pi(1)\in J(y)$	y	$s-2s_{\pi(1)}u^{\pi(1)}$	π	p	q
0	$p\ge1$	$n-1$	$\pi(1)\in I^{+}(y)\cup I^{-}(y)$	y	$s-2s_{\pi(1)}u^{\pi(1)}$	π	$p+1$	$q+1$

i	p	q	π	$\bar{y}$	$\bar{s}$	$\bar{\pi}$	$\bar{p}$	$\bar{q}$
0		$q<n-1$		y	$s-2s_{\pi(1)}u^{\pi(1)}$	π	p	q
$1\le i\le n$	0	n	$\pi(1)\in I^{+}(y)\cup I^{-}(y)$	y	$s-2s_{\pi(1)}u^{\pi(1)}$	π	p	$q-1$
			$\pi(1)\notin I^{+}(y)\cup I^{-}(y)$	y	$s-2s_{\pi(1)}u^{\pi(1)}$	π	p	q
	$i<p-1$			y	s	$(\pi(1),\ldots,\pi(i+1),$ $\pi(i),\ldots,\pi(n))$	p	q
	$i=p-1$			y	s	π	$p-1$	q
	$i>p-1$ $1\le p<n-1$			y	s	$(\pi(1),\ldots,\pi(p-1),$ $\pi(i),\pi(p),\ldots,$ $\pi(i-1),\pi(i+1),\ldots,$ $\pi(n))$	$p+1$	q
$n-1$	$n-1$			$y+2s_{\pi(n)}u^{\pi(n)}$	$s-2s_{\pi(n)}u^{\pi(n)}$	π	$q(\bar{\pi})-1$	$q(\bar{\pi})$
n				$y+2s_{\pi(n-1)}u^{\pi(n-1)}$	$s-2s_{\pi(n-1)}u^{\pi(n-1)}$	$(\pi(1),\ldots,\pi(n),$ $\pi(n-1))$	$q(\bar{\pi})-1$	$q(\bar{\pi})$

i	p	q	π	$\bar{y}$	$\bar{s}$	$\bar{\pi}$	$\bar{p}$	$\bar{q}$
$1 \le i \le n-q-2$		$2 \le q < n$		y	s	$(\pi(1),\ldots,\pi(i+1), \pi(i),\ldots,\pi(n))$	p	q
$n-q-1$	0		$\pi(n-q-1) \in T(y,s)$	y	s	$(\pi(1),\ldots,\pi(i+1), \pi(i),\ldots,\pi(n))$	p	$q+1$
	$p \ge 1$			y	s	$(\pi(1),\ldots,\pi(i+1), \pi(i),\ldots,\pi(n))$	$p+1$	$q+1$
			$\pi(n-q-1) \notin T(y,s)$	y	s	$(\pi(1),\ldots,\pi(i+1), \pi(i),\ldots,\pi(n))$	p	q
$n-q$	0			y	s	π	$p+1$	q
	1			y	s	π	$p-1$	q
	$p \ge 2$			y	s	$(\pi(1),\ldots,\pi(i+1), \pi(i),\ldots,\pi(n))$	$p-1$	$q-1$

i	p	q	π	$\bar{y}$	$\bar{s}$	$\bar{\pi}$	$\bar{p}$	$\bar{q}$
$n-q<i<n-q+p-1$	$p\geq 2$	$2\leq q<n$		y	s	$(\pi(1),\ldots,\pi(i+1),\pi(i),\ldots,\pi(n))$	p	q
$n-q+p-1$				y	s	π	$p-1$	q
$n-q+p-1<i$	$1\leq p<q-1$			y	s	π	$p+1$	q
$n-1$	$q-1$			$y+2s_{\pi(n)}u^{\pi(n)}$	$s-2s_{\pi(n)}u^{\pi(n)}$	π	$q(\bar{\pi})-1$	$q(\bar{\pi})$
n				$y+2s_{\pi(n-1)}u^{\pi(n-1)}$	$s-2s_{\pi(n-1)}u^{\pi(n-1)}$	$(\pi(1),\ldots,\pi(n),\pi(n-1))$	$q(\bar{\pi})-1$	$q(\bar{\pi})$
$n-q<i\leq n$	0			y	s	$(\pi(1),\ldots,\pi(n-q-1),\pi(i),\pi(n-q),\ldots,\pi(i-1),\pi(i+1),\ldots,\pi(n))$	p	$q-1$

i	p	q	π	$\bar{y}$	$\bar{s}$	$\bar{\pi}$	$\bar{p}$	$\bar{q}$
$1 \le i < n-2$		$q<2$		y	s	$(\pi(1),\ldots,\pi(i+1), \pi(i),\ldots,\pi(n))$	p	q
n				$y+2s_{\pi(n)}u^{\pi(n)}$	$s-2s_{\pi(n)}u^{\pi(n)}$	π	$q(\bar{\pi})-1$	$q(\bar{\pi})$
$n-2$			$\pi(i) \in T(y,s)$	y	s	$(\pi(1),\ldots,\pi(i+1), \pi(i),\ldots,\pi(n))$	p	$q+1$
			$\pi(i) \notin T(y,s)$	y	s	$(\pi(1),\ldots,\pi(i+1), \pi(i),\ldots,\pi(n))$	p	q

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